Journal of Applied Sciences, Information, and Computing
Vol. 3, No. 1 (June, 2022)
School of Mathematics and Computing, Kampala International University

ISSN: 1813-3509 https://jasic.kiu.ac.ug

Hadamard Product of a Class of Holomorphic Functions with an Arbitrary Fixed Point

BENJAMIN A. PETER[1], GEOFREY D. WANZALA[2], IDOWU A. SABIKI[3]

[1]Department of Mathematics and Statistics, Kampala International University, Ggaba Road, Kansanga, Kampala, Uganda, benjamin.aina@kiu.ac.ug
[2]Department of Sciences, Kampala International University, Ggaba Road, Kansanga, Kampala, Uganda, wanzaladeric8@gmail.com
[3]Department of Mathematical Sciences, Olabisi Onabanjo University, Ago, Nigeria, sabiq.idowu@ymail.com

Abstract

Hadamard product of holomorphic function is simply entry wise multiplication of two functions f and g in H. The Hadamard products of two functions have one thing in common that is, it involves the origin. Irrespective of the factors of the Hadamard product either power series or holomorphic functions, the open sets on which they are examined contain the origin. The aim of this study, therefore, is to investigate on the properties of Hadamard product for a class of holomorphic functions with an arbitrary fixed point. The concept of Hadamard product, Cauchy-Schwartz, holomorphic functions, Ruscheweyh differential operators, and Nevanlinna's theorem are employed in this study. This study generalized the coefficient inequalities for starlike and convex functions of exponential order β with an arbitrary fixed point using Ruscheweyh derivative. This study further provides an additional inequality and Hadamard product for a class of holomorphic functions with an arbitrary fixed point. It is concluded that Ruscheweyh derivative is an effective tool in the generalization of Hadamard product for a class of holomorphic functions with an arbitrary fixed point.

Keywords: Hadamard product, Arbitrary fixed point, Cauchy-Schwartz, holomorphic function, Nevanlinna's theorems.

1. Introduction

Hadamard product has been dated back to 1899 by J.S Hadamard, (1899). Hadamard product of holomorphic function is simply entry wise multiplication of two functions f and g.

The Hadamard products of two functions have one thing in common that is, it involves the origin. Irrespective of the factors of the Hadamard product either power series or holomorphic functions, the open sets on which they are examined contain the origin. Holomorphic function is a complex function that is differentiable at all points of an open set, i.e., analytic function or differentiable function. Holomorphic function implies that the function is analytic at a point which consequently admits a power series representation [1,2,3,4,5,6,7]. Hadamard product can also be defined as

$$(f_1 *,. \ . \ .* f_m)(z) = (z-\omega) + \sum_{q=n}^{\infty}\left(\prod_{j=1}^{m} a_{q,j}\right)(z-\omega)^q . \quad (1)$$

Let H_0 be the class of functions $f(z)$ that is analytic in the unit disc. Let $D := \{z : |z| < 1\}$ normalized by $f(0) = 0$ and $f'(0) = 1$.
The Hadamard product of two functions

$f(z) = z + \sum_{k=2}^{\infty} a_k z^k$ and $g(z) = z + \sum_{k=2}^{\infty} b_k z^k$

in H_0 is given as:

$$(f * g)(z) = z + \sum_{k=2}^{\infty} a_k b_k z^k \qquad (2)$$

Acu and Owa (2005) examined on a subclass of $n-$starlike functions. The subclass is related to functions with positive real part and defined classes of close to convex functions with the usual normalization $f(\omega) = f'(\omega) - 1 = 0, \mathrm{Re}((z-\omega)f'(z)/f(z)) > 0, z \in U$. Darus and Owa (2017) studied new subclasses concerning some analytic and univalent functions. They employed series expansion in an open unit disk D. Some partial sums of $f(z)$ and analytic functions are derived. Also new subclasses of $f(z)$ that are analytic and univalent are obtained. Faisal and Darus (2017) studied subclasses of analytic functions. They employed new linear fractional differential operator technique. Convolution properties of some subclasses of functions and inclusion relationships are established. Mustapha (2017) considered characteristic properties of the new subclasses of analytic functions. He used Ruscheweyh Derivative method. The properties of new subclasses of analytic functions and several coefficient inequalities of subclasses of analytic functions are derived. Oladipo (2015) studied coefficient estimates for some families of analytic univalent functions associated with $q-$analogue of Dziok-Srivastava operator. He employed Cauchy-Schwartz inequality techniques. Several consequences of new subclass of bi-univalent functions in the open unit disk is derived.

$$\mathrm{Re}((z-\omega)f'(z)/f(z)) > \beta, \; z \in E.$$

In recent times, the study of geometric functions, convolution, and coefficient inequalities has been studied by many researchers due to its proper and efficient analysis of function [Sokół (2006) & Ukeje and Nnadi (2012)]. Hadamard product or coefficient inequalities are known for some subclasses of functions, it is expedient to determine the same for a large class of family. However, this necessitates the study of Hadamard product in this research work for a large class of holomorphic function with arbitrary fixed point and to determine the coefficient inequalities of exponential order β and also to improve on the bounds when the class satisfy Hadamard product. This study improved on [Oladipo (2015) & Ukeje and Nnadi (2012)] by incorporating holomorphic function of exponential order β.

2 Materials and Methods

In this paper, generalization of the coefficient inequalities of a class of holomorphic functions considered using Ruscheweyh Derivative, generalized hypergeometric functions and the convolution properties of a class of functions with arbitrary fixed point.

3 Model Formulation

3.1 Lemma (3.1)

A function $f(z) \in A(\omega)$ is contained in $S^*(\omega, \beta)$ if and only if

$$\sum_{k=n}^{\infty} (k + e^{\beta} - 2)(r+d)^{k-1} a_k \le 1 - e^{\beta} \qquad (3)$$

where $|z| = r < 1$ and $|\omega| = d$.

Extremal function denoted by

$$f(z) = (z-\omega) + \frac{1-e^{\beta}}{(k+e^{\beta}-2)(r+d)^{k-1}}(z-\omega)^{k-1}, k \ge 2 \qquad (4)$$

Proof

Suppose the inequality in (48) holds, $|z| < 1$ and $|\omega| = d$. Then,

$$\left|\frac{(z-\omega)f'(z)}{f(z)} - 1\right| = \left|\frac{\sum_{k=n}^{\infty}(k-1)a_k(z-\omega)^{k-1}}{1+\sum_{k=n}^{\infty}a_k(z-\omega)^{k-1}}\right| \le \frac{\sum_{k=n}^{\infty}(1-1)(r+d)^{k-1}a_k}{1+\sum_{k=n}^{\infty}(r+d)^{k-1}a_k} \le 1 - e^{\beta} \qquad (5)$$

It implies that $\frac{(z-\omega)f'(z)}{f(z)}$ remains in the center of the circle ω whose radius is $1-e^{\beta}$. For this reason, $f(z)$ is within the class $S^*(\omega, e^{\beta})$.

Conversely, suppose $f(z)$ is defined by (49) is within the class $S^*(\omega, e^{\beta})$. Then,

$$\text{Re}\left(\frac{(z-\omega)f'(z)}{f(z)}\right) = \text{Re}\left(\frac{1+\sum_{k=n}^{\infty} ka_k (z-\omega)^{k-1}}{1+\sum_{k=n}^{\infty} a_k (z-\omega)^{k-1}}\right) > e^{\beta}$$

(6)

For $z \in D$. Pick a real number of z, which implies that $\frac{(z-\omega)f'(z)}{f(z)}$ is a real number. Assuming that z approaching 1^- equation (6) yields

$$e^{\beta}\left(1+\sum_{k=2}^{\infty}(r+d)^{k-1}a_k\right) \le 1+\sum_{k=2}^{\infty}k(r+d)^{k-1}a_k \quad (7)$$

This completes the proof of Lemma 3.1

Lastly, the postulate of Lemma 3.1 is sharp with the extremal of

$$f(z) = (z-\omega) + \frac{1-e^{\beta}}{(k+e^{\beta}-2)(r+d)^{k-1}}(z-\omega)^{k-1}, k \ge 2 \quad (8)$$

3.1.1 Theorem (3.1)

Let $D^n f(z)$ be a given function on $j = 1,2$ such that $D^n f(z) \in A(\omega)$ is a subclass $S^*(\omega, e^{\beta}, n)$ if and only if by

$$\sum_{k=q}^{\infty} \psi_{m,n}\left(k+kn+kn^2+e^{\beta_j}-2\right)(r+d)^{k-1}a_{k,j} \le 1-e^{\beta_j}$$

it follows that

$$\sum_{k=q}^{\infty} \psi_{m,n}\left(k+kn+kn^2+e^{\beta_j}-2\right)(r+d)^{k-1}a_{k,j} \le 1-e^{\beta_j}$$

(9)

where

$$\psi_{n,m} = \frac{(m+n-1)!}{n!(m-1)!} \quad (10)$$

$|z| = r < 1$ and $|\omega| = d$.

Proof

Let $D^n f(z) \in S^*(\omega, e^{\beta}, n)$ with $|z| = r < 1$ and $|\omega| = d$ then

$$\frac{(z-\omega)\left(D^n f_j(z)\right)'}{D^n f_j(z)} = \frac{(z-\omega)+\sum_{k=q}^{\infty}\psi_{n,m}\frac{\left(k+kn+kn^2-1\right)n!(\mathrm{m}-1)!\,a_{k,j}(z-\omega)^k}{(\mathrm{m}+n)!}}{(z-\omega)+\sum_{k=q}^{\infty}\frac{n!(\mathrm{m}-1)!}{(\mathrm{m}+n-1)!}a_{k,j}(z-\omega)^k} \tag{11}$$

and

$$\mathrm{Re}\left(\frac{(z-\omega)\left(D^n f_j(z)\right)'}{D^n f_j(z)}\right) = \left|\frac{(z-\omega)\left(D^n f_j(z)\right)'}{D^n f_j(z)}-1\right| \tag{12}$$

$$\leq \left|\frac{\left(\frac{\sum_{k=q}^{\infty}\psi_{n,m}\left(k+kn+kn^2-1\right)n!(\mathrm{m}-1)!\,a_{k,j}(z-\omega)^k}{(\mathrm{m}+n)!}\right)-\sum_{k=q}^{\infty}\psi_{n,m}\frac{n!(m-1)!}{(\mathrm{m}+n-1)!}a_{k,j}(z-\omega)^k}{(z-\omega)+\sum_{k=q}^{\infty}\psi_{n,m}\frac{n!(m-1)!}{(\mathrm{m}+n-1)!}a_{k,j}(z-\omega)^k}\right| \tag{13}$$

$$\leq -\left(\frac{\beta_1}{1}+\frac{(\beta_2)^2}{2}\right)$$

Since $n(n-1)(n-2)\ldots(2)+m(m-1)(m-2)\ldots 2 > m(m-1)(m-2)\ldots 2+(n-1)(n-2)\ldots 2$

It implies that

$$\frac{1}{n(n-1)(n-2)\ldots(2)+m(m-1)(m-2)\ldots 2} < \frac{1}{m(m-1)(m-2)\ldots 2+(n-1)(n-2)\ldots 2} \tag{14}$$

Equations (13)-(14) yield

$$\frac{\sum_{k=q}^{\infty}\psi_{n,m}\frac{n!(\mathrm{m}-1)!}{(\mathrm{m}+n-1)!}a_{k,j}(r+d)^{k-1}\left(k+kn+kn^2-1\right)}{1+\sum_{k=q}^{\infty}\psi_{n,m}\frac{n!(\mathrm{m}-1)!}{(\mathrm{m}+n-1)!}a_{k,j}(r+d)^{k-1}} \leq -\left(\frac{\beta_1}{1}+\frac{(\beta_2)^2}{2}\right) \tag{15}$$

Substituting for $\psi_{n,m}$ in equation (15) it yields

$$\sum_{k=q}^{\infty}\psi_{n,m}\left(k+kn+kn^2+\frac{\beta_j}{1}+\frac{(\beta_j)^2}{2}-1\right)(r+d)^{k-1}a_{k,j} \leq -\left(\frac{\beta_j}{1}+\frac{(\beta_j)^2}{2}\right) \tag{16}$$

The result in equation (16) is sharp for the function given by Ruscheweyh derivative of order n, given as

$$D^n f_j(z) = (z-\omega) + \frac{1-e^{\beta_j}}{\psi_{n,m}\left(kn+k+kn^2+e^{\beta_j}-2\right)}(z-\omega)^k \tag{17}$$

3.2 Theorem (3.2)

Let $f(z)$ be a function $f_j(z)$ on $j=1,2.$ Then, $D^n f_j(z) \in A(\omega)$ is a subclass $S^c\left(\omega, e^{\beta_j}, n\right)$ if and only if

$$\sum_{k=q}^{\infty} \psi_{n,m}\left(k+kn+kn^2-n+e^{\beta_j}-2\right)(r+d)^{k-1} a_{k,j} \leq 1-e^{\beta_j} \tag{18}$$

Proof

Suppose that $D^n f_j(z) \in S^c\left(\omega, e^{\beta}, n\right)$ then

$$1+\frac{(z-\omega)\left(D^n f_j(z)\right)''}{\left(D^n f_j(z)\right)'} = \frac{\sum_{k=q}^{\infty}\left(\frac{k(k-1)(n+1)!(\mathrm{m}-1)!\, a_{k,j}}{(\mathrm{m}+n)!}\right)(z-\omega)^{k-1}}{1+\sum_{k=q}^{\infty}\left(\frac{kkn!(\mathrm{m}-1)!}{(\mathrm{m}+n-1)!}\right)(z-\omega)^{k-1}}+1 \tag{19}$$

Consequently,

$$\mathrm{Re}\left(1+\frac{(z-\omega)\left(D^{n+1} f_j(z)\right)''}{D^n f_j(z)'}\right)$$

$$\leq \frac{\sum_{k=q}^{\infty}\left(\frac{k(k-1)(n+1)(m-1)!\, a_{k,j}}{(\mathrm{m}+1)!}\right)(r+d)^{k-1}+\sum_{k=q}^{\infty}\left(\left(\frac{kkn!(m-1)!}{(\mathrm{m}+n-1)!}\, a_{k,j}\right)(r+d)^{k-1}\right)}{1+\sum_{k=q}^{\infty}\left(\frac{kn!(m-1)!}{(\mathrm{m}+n-1)!}\, a_{k,j}\right)(r+d)^{k-1}}$$

$$\leq -\left(\frac{\beta_1}{1}+\frac{(\beta_2)^2}{2}\right) \tag{20}$$

Equations (18) -(20) yield

$$\operatorname{Re}\left(1+\frac{(z-\omega)\left(D^{n+1}f_j(z)\right)''}{\left(D^n f_j(z)\right)'}\right)\le\frac{\sum_{k=q}^{\infty}k\left(k+kn-n+n^2+e^{\beta_j}-2\right)(r+d)^{k-1}a_{k,j}}{1+\sum_{k=q}^{\infty}\psi_{n,m}k(r+d)^{k-1}a_{k,j}}\le-\left(\frac{\beta_1}{1}+\frac{(\beta_2)^2}{2}\right) \tag{21}$$

Simplifying equation (21) further it yields

$$\sum_{k=q}^{\infty}k\left(k+kn-n+n^2+\frac{\beta_1}{1}+\frac{(\beta_2)^2}{2}-1\right)(r+d)^{k-1}a_{k,j}\le-\left(\frac{\beta_1}{1}+\frac{(\beta_2)^2}{2}\right)\left(1+\sum_{k=q}^{\infty}\psi_{n,m}k(r+d)^{k-1}a_{k,j}\right) \tag{22}$$

The extremal function is given below as

$$D^n f_j(z)=(z-\omega)+\frac{-\left(\frac{\beta_1}{1}+\frac{(\beta_2)^2}{2}\right)}{\psi_{n,m}\left[k\left(k+kn-n+n^2-1+\left(\frac{\beta_1}{1}+\frac{(\beta_2)^2}{2}\right)\right)\right]}(z-\omega)^k \tag{23}$$

3.3 Theorem (3.3)

Let $D^n f_j(z)\in A(\omega)$ be a subclass in $S^*\left(\omega,e^{\beta},n\right),\ j=(1,\ldots,m)$. Then $D^n\left(f_1*\ldots*f_m\right)\in S^*(\omega,\delta,n)$

where

$$\delta=1-\frac{\left(kn+kn^2+k-2\right)\prod_{j=1}^{m}\left(1-e^{\beta_j}\right)}{\prod_{j=1}^{m}\left(1-e^{\beta_j}\right)+\prod_{j=1}^{m}\left(kn+kn^2+k-2\right)(r+d)^{k-1}}$$

$$D^n f_j(z)=(z-\omega)+\frac{\left(1-e^{\beta_j}\right)}{\left(kn+k+kn^2+e^{\beta_j}-2\right)}(z-\omega)^k$$

Proof

Following the techniques of [1]

Let $D^n f_1(z)$ belong to $\left(\omega,e^{\beta_1},n\right)$ and $D^n f_2(z)$ belong to $\left(\omega,e^{\beta_2},n\right)$. Then, the inequality in Theorem 3.1 implies that

$$\sum_{k=q}^{\infty}\sqrt{\left(\frac{\psi_{n,m}\left(k+kn+kn^2+e^{\beta_j}-2\right)(r+d)^{k-1}}{1-e^{\beta_j}}\right)}a_{k,j}\le 1 \tag{24}$$

Following the application of Cauchy-Schwartz inequality it yields

$$\left|\sum_{k=q}^{\infty}\sqrt{\left(\frac{\psi_{n,m}^{2}\left(k+kn+kn^{2}+e^{\beta_j}-2\right)\left(r+d\right)^{k-1}}{\left(1-e^{\beta_1}\right)\left(1-e^{\beta_2}\right)}\right)a_{k,1},a_{k,2}}\right|^{2}$$

$$\leq\left(r+d\right)^{k-1}\left[\sum_{k=q}^{\infty}\frac{\psi_{n,m}\left(k+kn+kn^{2}+e^{\beta_1}-2\right)}{\left(1-e^{\beta_1}\right)}a_{k,1}\right]\left[\sum_{k=q}^{\infty}\frac{\psi_{n,m}\left(k+kn+kn^{2}+e^{\beta_2}-2\right)}{\left(1-e^{\beta_2}\right)}a_{k,2}\right]\leq 1 \quad (25)$$

It implies that

$$\sum_{k=q}^{\infty}\frac{\psi_{n,m}\left(k+kn+kn^{2}+\lambda-2\right)}{\left(1-\lambda\right)}a_{k,1},a_{k,2}\leq\sum_{k=q}^{\infty}\sqrt{\frac{\psi_{n,m}^{2}\left(k+kn+kn^{2}+e^{\beta_j}-2\right)\left(k+kn+kn^{2}+e^{\beta_2}-2\right)}{\left(1-e^{\beta_1}\right)\left(1-e^{\beta_2}\right)}}a_{k,1},a_{k,2} \quad (26)$$

$$\sqrt{a_{k,1},a_{k,2}}\leq\frac{\left(1-\lambda\right)}{\left(k+kn+kn^{2}+\lambda-2\right)}\sqrt{\frac{\left(k+kn+kn^{2}+e^{\beta_1}-2\right)\left(k+kn+kn^{2}+e^{\beta_2}-2\right)}{\left(1-e^{\beta_1}\right)\left(1-e^{\beta_2}\right)}} \quad (27)$$

It follows that if $D^{n}\left(f_1 * f_2\right)(z)$ belong to $S^{*}\left(\omega,\lambda,n\right)$ and the inequality yields

$$\sqrt{a_{k,j}}\leq\sqrt{\frac{\left(1-e^{\beta_j}\right)}{\left(k+kn+kn^{2}+e^{\beta_j}-2\right)}\left(r+d\right)^{k-1}}\qquad\left(j=1,2: k=q,q+1,q+2,\ldots\right) \quad (28)$$

For this reason if

$$\frac{\left(1-e^{\beta_1}\right)\left(1-e^{\beta_2}\right)}{\left(k+kn+kn^{2}+e^{\beta_1}-2\right)\left(k+kn+kn^{2}+e^{\beta_2}-2\right)\left(r+d\right)^{k-1}}$$

$$\leq\frac{\left(1-\lambda\right)}{\left(k+kn+kn^{2}+\lambda-1\right)}\sqrt{\frac{\left(k+kn+kn^{2}+e^{\beta_1}-2\right)\left(k+kn+kn^{2}+e^{\beta_2}-2\right)}{\left(1-e^{\beta_1}\right)\left(1-e^{\beta_2}\right)}} \quad (29)$$

Because of this, we obtain

$$\frac{\left(k+kn+kn^{2}+\lambda-2\right)}{\left(1-\lambda\right)}\leq\frac{\left(k+kn+kn^{2}+e^{\beta_1}-2\right)\left(k+kn+kn^{2}+e^{\beta_2}-2\right)\left(r+d\right)^{k-1}}{\left(1-e^{\beta_1}\right)\left(1-e^{\beta_2}\right)} \quad (30)$$

Thus, if $D^{n}\left(f_1 * f_2\right)(z)$ belong to $S^{*}\left(\omega,\lambda,n\right)$ then

$$\lambda\leq 1-\frac{\left(k+kn+kn^{2}+\lambda-2\right)\left(1-e^{\beta_1}\right)\left(1-e^{\beta_2}\right)}{\left(1-e^{\beta_1}\right)\left(1-e^{\beta_2}\right)+\left(k+kn+kn^{2}+e^{\beta_j}-1\right)\left(r+d\right)^{k-1}}=L(k),\quad\left(k=q,q+1,\ldots\right) \quad (31)$$

Hence, $L(k)$ is increasing for $k>n,$ it then yields

$$\lambda = 1 - \frac{\left(q + qn + qn^2 - 1\right)\left(1 - e^{\beta_1}\right)\left(1 - e^{\beta_2}\right)}{\left(1 - e^{\beta_1}\right)\left(1 - e^{\beta_2}\right) + \left(q + qn + qn^2 + e^{\beta_1} - 2\right)\left(q + qn + qn^2 + e^{\beta_2} - 2\right)\left(r + d\right)^{(k-1)2}} \tag{32}$$

Assume that $D^n\left(f_1 * f_2\right)(z)$ belong to $S^*\left(\omega, \alpha, n\right)$

where

$$\alpha = \frac{\left(k + kn + kn^2 - 1\right)\prod_{j=1}^{m}\left(1 - e^{\beta_j}\right)}{\left(r + d\right)^{k-1}\prod_{j=1}^{m}\left(1 - e^{\beta_j}\right)\left(q + qn + qn^2 + e^{\beta_j} - 2\right) + \prod_{j=1}^{m}\left(1 - e^{\beta_j}\right)},$$

It implies that if $D^n\left(f_1 * f_2\right)(z)$ belong to $S^*\left(\omega, \delta, n\right)$

Then,

$$\delta = 1 - \frac{\left(k + kn + kn^2 - 1\right)\prod_{j=1}^{m+1}\left(1 - e^{\beta_1}\right)\left(1 - e^{\beta_2}\right)}{\left(r + d\right)^{k-1}\prod_{j=1}^{m+1} + \left(k + kn + e^{\beta_1} - 2\right)\left(k + kn + e^{\beta_2} - 2\right)\left(r + d\right)^{k-1}} \tag{33}$$

Therefore

$$D^n\left(f_1 * \ldots * f_m\right)(z) = (z - \omega) + \left(\prod_{j=1}^{m}\left(\frac{\left(1 - e^{\beta_J}\right)}{\left(r + d\right)^{k-1}\left(k + kn + e^{\beta_J} - 2\right)}\right)\right)(z - \omega)^k \tag{34}$$

This completes the proof of Theorem 3.3.

3.4 Theorem (3.4)

If $D^n f_j(z) \in S^c\left(\omega, e^{\beta}, n\right)(j = 1, \ldots, m)$ subsequently $D^n\left(f_1 * \ldots * f_m\right)(z) \in S^c\left(\omega, \lambda, n\right)$

where

$$\lambda = 1 - \frac{\left(kn + k + kn^2 - n - 1\right)\prod_{j=1}^{m} - \left(\frac{\beta_1}{1} + \frac{\left(\beta_2\right)^2}{2}\right)}{\psi_{n,m} q^{m-1}\left(r + d\right)^{k-1}\coprod_{=1}^{m}\left(kn + k + kn^2 - n - 2\right) + \prod_{j=1}^{m} - \left(\frac{\beta_1}{1} + \frac{\left(\beta_2\right)^2}{2}\right)}$$

$$D^n f_j(z) = (z - \omega) + \frac{-\left(\frac{\beta_1}{1} + \frac{\left(\beta_2\right)^2}{2}\right)}{\psi_{n,m}\left(\left(kn + k + kn^2 + e^{\beta_j} - 2\right)\right)}(z - \omega)^k \tag{35}$$

Proof

For $D^n f_j(z)$ belong to $S^c(\omega, e^{\beta}, n)(j = 1,2,...,m)$ the inequalities yield

$$\sum_{k=q}^{\infty} \sqrt{\frac{\psi_{n,m}\left(k\left(k+kn+n^2-n+e^{\beta_1}-2\right)\right)\psi_{n,m}\left(k\left(k+kn+n^2-n+e^{\beta_2}-2\right)\right)}{\left(1-e^{\beta_1}\right)\left(1-e^{\beta_2}\right)}(r+d)^{k-1} a_{k,1} a_{k,2}} \le 1 \tag{36}$$

It implies that $D^n(f_1 * f_2)(z)$ belong to $S^*(\omega, \lambda, n)$ then

Following the Cauchy-Schwartz inequality it yields

$$\left| \sum_{k=q}^{\infty} \sqrt{\frac{\psi_{n,m}\left(k\left(k+kn+n^2+e^{\beta_1}-n-2\right)\right)\psi_{n,m}\left(k\left(k+kn+n^2+e^{\beta_2}-n-2\right)\right)(r+d)^{k-1} a_{k,1} a_{k,2}}{\left(1-e^{\beta_1}\right)\left(1-e^{\beta_2}\right)}} \le 1 \right|^2$$

$$\le (r+d)^{k-1} \sum_{k=q}^{\infty} \left(\frac{\psi_{n,m}\left(k\left(k+kn+n^2+e^{\beta_1}-n-2\right)\right)}{\left(1-e^{\beta_1}\right)} \right) \sum_{k=q}^{\infty} \left(\frac{\psi_{n,m}\left(k\left(k+kn+n^2+e^{\beta_2}-n-2\right)\right)}{\left(1-e^{\beta_2}\right)} \right) a_{k,2} \le 1 \tag{37}$$

Similarly, if

$$\sum_{k=q}^{\infty} \left(\frac{\psi_{n,m}^2\left(k\left(k+kn+n^2+\lambda-n-2\right)\right)}{(1-\lambda)} \right) a_{k,1} a_{k,2}$$

$$\sum_{k=q}^{\infty} \frac{\psi_{n,m}\left(k\left(k+kn+kn^2+e^{\beta_1}-n-2\right)\right)\psi_{n,m}\left(k\left(k+kn+kn^2+e^{\beta_2}-n-2\right)\right)(r+d)^{k-1} a_{k,1} a_{k,2}}{\left(1-e^{\beta_1}\right)\left(1-e^{\beta_2}\right)} \tag{38}$$

Such that

$$\sqrt{a_{k,1} a_{k,2}} \le \left[\frac{1-\lambda}{\psi_{n,m}^2\left(k\left(k+kn+n^2+\lambda-n-1\right)\right)} \right] * \sqrt{\frac{\psi_{n,m}\left(k\left(k+kn+n^2+e^{\beta_1}-n-2\right)\right)\psi_{n,m}\left(k\left(k+kn+n^2+e^{\beta_2}-n-2\right)\right)(r+d)^{k-1} a_{k,1} a_{k,2}}{\left(1-e^{\beta_1}\right)\left(1-e^{\beta_2}\right)}}$$

$$= \left[\frac{1-\lambda}{\psi_{n,m}^2\left(k\left(k+kn+n^2+\lambda-n-2\right)\right)} \right] \sqrt{\frac{\psi_{n,m}^2 \prod_{j=1}^{2}\left(k\left(k+kn+n^2+e^{\beta_j}-n-2\right)\right)(r+d)^{k-1}}{\prod_{j=1}^{2}\left(1-e^{\beta_j}\right)}} \tag{39}$$

Then, equation (39) shows that $D^n(f_1 * f_2)(z)$ belong to $S^*(\omega, \lambda, n)$ then

$$\sqrt{a_{k,1}a_{k,2}} \leq \left[\frac{\prod_{j=1}^{2}\left(1-e^{\beta_j}\right)}{\psi_{n,m}^{2}\prod_{j=1}^{2} k\left(k+kn+n^{2}+e^{\beta_j}-n-2\right)\left(r+d\right)^{k-1}} \right] \tag{40}$$

Thus if

$$\left[\frac{\prod_{j=1}^{2}\left(1-e^{\beta_j}\right)}{\psi_{n,m}^{2}\prod_{j=1}^{2} k\left(k+kn+n^{2}+e^{\beta_j}-n-2\right)\left(r+d\right)^{k-1}} \right]$$

$$\leq \left[\frac{1-\lambda}{\psi_{n,m}^{2}\left(k\left(k+kn+n^{2}+\lambda-n-1\right)\right)} \right] \sqrt{\frac{\psi_{n,m}^{2}\prod_{j=1}^{2}\left(k\left(k+kn+n^{2}+e^{\beta_j}-n-2\right)\right)\left(r+d\right)^{k-1}}{\prod_{j=1}^{2}\left(1-e^{\beta_j}\right)}} \tag{41}$$

Hence,

$$\frac{\left(k\left(k+kn+n^{2}+\lambda-n-2\right)\right)}{1-\lambda} \leq \frac{\psi_{n,m}^{2}\prod_{j=1}^{2}\left(k\left(k+kn+n^{2}+e^{\beta_j}-n-1\right)\right)\left(r+d\right)^{k-1}}{\prod_{j=1}^{2}\left(1-e^{\beta_j}\right)}, \ (k=q,\ q+1,...) \tag{42}$$

Consequently, $D^{n}\left(f_1 * f_2\right)(z)$ belong to $S^{c}\left(\omega,\lambda,n\right)$, equation (42) becomes

$$\lambda = 1-\frac{k\left(k+kn+n^{2}-n-2\right)\left(1-e^{\beta_1}\right)\left(1-e^{\beta_2}\right)}{k\left(1-e^{\beta_1}\right)\left(1-e^{\beta_2}\right)+k\left(k+kn+n^{2}-n+e^{\beta_1}-2\right)\left(k+kn+n^{2}-n+e^{\beta_2}-2\right)\left(r+d\right)^{k-1}},\ k=q,q+1,... \tag{43}$$

Since equation (39) is an increasing function, it implies that $k \geq q$ which shows that $D^{n}\left(f_1 * f_2\right)(z)$ belong to $S^{*}\left(\omega,\Lambda,n\right)$

Then

$$\Lambda = 1-\frac{\left(q+qn+qn^{2}-n-1\right)\prod_{j=1}^{m}\left(1-e^{\beta_j}\right)}{q^{m-1}\prod_{j=1}^{2}\left(q+qn+qn^{2}+e^{\beta_j}-n-2\right)\left(r+d\right)^{k-1}+\prod_{j=1}^{m+1}\left(1-e^{\beta_j}\right)} \tag{44}$$

In the same vein, if $D^{n}\left(f_1 * ... * f_{m+1}\right)(z)$ belong to $S^{c}\left(\omega,\delta,n\right)$, then

$$\delta = 1-\frac{\left(q+qn+qn^{2}-n-1\right)\prod_{j=1}^{m+1}\left(1-e^{\beta_j}\right)}{q^{m}\prod_{j=1}^{m+1}\left(q+qn+qn^{2}+e^{\beta_j}-n-2\right)\left(r+d\right)^{k-1}+\prod_{j=1}^{m+1}\left(1-e^{\beta_j}\right)} \tag{45}$$

This completes the proof of Theorem 3.4.

3.5 Theorem (3.5)

Suppose $f(z)=(z-\omega)+\sum_{n=2}^{\infty}|a_n|(z-\omega)^n$ and

$$H\left(\alpha_1,\ldots,\alpha_k:-\left(\frac{\beta_1}{1}+\frac{(\beta_2)^2}{2}\right)\right)f(z)=h\left(\alpha_1,\ldots,\alpha_k:-\left(\frac{\beta_1}{1}+\frac{(\beta_2)^2}{2}\right):z-\omega\right)*f(z) \quad (46)$$

Then

$$H\left(\alpha_1,\ldots,\alpha_k:-\left(\frac{\beta_1}{1}+\frac{(\beta_2)^2}{2}\right)\right)f(z)=(z-\omega)+\sum_{n=2}^{\infty}\Gamma_n^{-1}a_n(z-\omega)^n \quad (47)$$

where

$$\Gamma_n=\frac{-\left(\frac{\beta_1}{1}+\frac{(\beta_2)^2}{2}\right)_{n-1}}{(\alpha_1)_{n-1}\ldots(\alpha_k)_{n-1}}(n-1)(n-2)\ldots2$$

Proof

Suppose

$$h\left(\alpha_1,\ldots,\alpha_k:-\left(\frac{\beta_1}{1}+\frac{(\beta_2)^2}{2}\right)\right)(z-\omega)$$

$$=(z-\omega)\frac{(\alpha_1)_0\ldots(\alpha_k)_0}{-\left(\frac{\beta_1}{1}+\frac{(\beta_2)^2}{2}\right)_0}+\frac{(\alpha_2)_1\ldots(\alpha_k)_1}{-\left(\frac{\beta_1}{1}+\frac{(\beta_2)^2}{2}\right)_1}(z-\omega)+\ldots \quad (48)$$

Satisfying the condition for Hadamard product and

$$H\left(\alpha_1,\ldots,\alpha_k:-\left(\frac{\beta_1}{1}+\frac{(\beta_2)^2}{2}\right)\right)f(z)=(z-\omega)+\sum_{n=2}^{\infty}\frac{(\alpha_1)_{n-1}\ldots(\alpha_k)_{n-1}}{-\left(\frac{\beta_1}{1}+\frac{(\beta_2)^2}{2}\right)_{n-1}}\frac{a_n(z-\omega)^n}{(n-1)(n-2)\ldots2} \quad (49)$$

Equations (37) and (39) yield

$$\Gamma_n^{-1}=\frac{(\alpha_1)_{n-1}\,.\,.\,.\,(\alpha_{k-1})_{n-1}}{\left(\frac{\beta_1}{1}+\frac{(\beta_2)^2}{2}\right)_{n-1}}(n-1)(n-2)\,.\,.\,.\,2 \quad (50)$$

which means that

$$\Gamma_n = \frac{-\left(\frac{\beta_1}{1}+\frac{(\beta_2)^2}{2}\right)_{n-1}}{(\alpha_1)_{n-1}\ldots(\alpha_k)_{n-1}}(n-1)(n-2)\ldots 2 \tag{51}$$

This completes the proof of Theorem 3.5

3.6 Theorem (3.6)

Suppose $f(z)=(z-\omega)+e^{i\theta}\sum_{n=2}^{\infty}|a_n|(z-\omega)^n$ and $g(z)=(z-\omega)+e\sum_{k=n}^{\infty}b_n(z-\omega)^k$ are analytic in D

where $|z|=r<1$ and $|\omega|=d$

Given that $f(z)<g(z)$ and $g(z)$ belong to $S^c(\omega)$ then

$$|a_n|(r+d)^{n-1}\leq 1 \tag{52}$$

Proof

Given that $f<g(z)$ then $f(z)=g(\omega(z))$ and $|\omega(z)|\leq 1$ satisfying $f(z)\leq g(z)$

Assume that $g(z)$ belong to $S^c(\omega)$ if and only if $f(z)=(z-\omega)g'(z)$ belong to the family of starlike functions where ω is the arbitrary fixed point. Hence, following the applications of Nevanlinna's theorem on the starlike functions and the implications of subordination of $f<g(z)$ then $f(z)=g(\omega(z))$ and $|\omega(z)|\leq 1$ satisfying $f(z)\leq g(z)$.

Nevanlinna's theorem and Durren's subordination implications further yield

$$|nb_n|(r+d)^{n-1}=|na_n|(r+d)^{n-1}\leq n \tag{53}$$

Equation (53) consequently yields

$$|a_n|(r+d)\leq 1 \tag{54}$$

This completes the proof 3.6.

4. Results and Discussion

Corollaries 4.1 to 4.4 are the consequences of Lemma 3.1 and Theorem 3.1

Corollary 4.1

Let $f(z)$ belong to $A(\omega)$ exist within the class $S^*(\omega, e^{\beta})$. Then,

$$a_k \leq \frac{1-e^{\beta}}{\left(k+e^{\beta}-2\right)\left(r+d\right)^{k-1}}, k \geq 2 \tag{55}$$

where $|\omega| = d$, the inequality in (55) remains true for $f(z)$ stated by (8)

Corollary 4.1.1

A function $D^n f_j(z) \in A(\omega)$ is in the class $S^*\left(\omega, e^{\beta}, 0\right)$, if and only if

$$\sum_{k=p}^{\infty}\left(k+e^{\beta_j}-2\right)\left(r+d\right)^{k-1} a_k \leq 1-e^{\beta_j} \tag{56}$$

where $|z| = r < 1, |\omega| = d, n = 0$

Corollary 4.2

Let $D^n f_j(z) \in A(\omega)$ belong to the class $S^*\left(\omega, e^{\beta}, n\right)$

$$a_{k,j} \leq \frac{1-e^{\beta_j}}{\psi_{0,m}\left(k+e^{\beta_j}-2\right)\left(r+d\right)^{k-1}}, k \geq 2 \tag{57}$$

where $|z| = r < 1, |\omega| = d$ and the inequality in (57) is valid for functions given in (9)

Corollary 4.3

Suppose a function $D^n f(z) \in A(\omega)$ is contained in $S^*\left(\omega, e^{\beta}, 0\right)$, then

$$a_{k,j} \leq \frac{1-e^{\beta_j}}{\psi_{0,\mathrm{m}}\left[k\left(kn-n+k+e^{\beta_j}-2\right)\left(r+d\right)^{k-1}\right]} k \geq 2, \psi_{0,\mathrm{m}} = 1 \tag{58}$$

where $|z| = r < 1$, and $|\omega| = d$ where

The equality in (58) is valid for functions given in (17)

This completes the consequences of Theorem 3.1.

The consequences of Theorem 3.2 are the following corollaries (3.4)-(3.7)

Corollary 4.4

Let $$f_j(z) = (z-\omega) + \left(\frac{1-e^{\beta_j}}{\left(n+e^{\beta_j}-2\right)\left(r+d\right)^{k-1}}\right)(z-\omega)^n, \left(n = 2,3,.. \quad j = 1,\ldots,r\right) \tag{59}$$

and

$$g_j(z)=(z-\omega)+\left(\frac{1-e^{\alpha_j}}{n\left(n+e^{\beta_j}-2\right)(r+d)^{k-1}}\right)(z-\omega)^n,\ (n=2,3,..\quad j=1,...,s) \tag{60}$$

Corollary 4.5

Suppose a function $D^n f(z)\in A(\omega)$ is contained in $S^*\left(\omega,e^{\beta},0\right)$, then

$$a_{k,j}\le\frac{1-e^{\beta_j}}{\psi_{n,\mathrm{m}}\left[k\left(kn-n+k+e^{\beta_j}-2\right)(r+d)^{k-1}\right]} \tag{61}$$

where $|z|=r<1$, and $|\omega|=d$

The inequality holds for equation (17).

Corollary 4.6

Suppose a function $D^n f(z)\in A(\omega)$ is contained in $S^*\left(\omega,e^{\beta},n\right)$, then

$$a_{k,j}\le\frac{1-e^{\beta_j}}{\psi_{n,\mathrm{m}}\left[k\left(kn-n+k+e^{\beta_j}-2\right)(r+d)^{k-1}\right]} \tag{62}$$

where $|z|=r<1$ and the inequality holds for equation (17)

Corollary 4.7

Suppose a function $D^n f(z)\in A(\omega)$ is contained in $S^*\left(\omega,e^{\beta},0\right)$, then

$$a_{k,j}\le\frac{1-e^{\beta_j}}{\psi_{0,\mathrm{m}}\left[k\left(k+e^{\beta_j}-2\right)(r+d)^{k-1}\right]},\psi_{0,\mathrm{m}}=1 \tag{63}$$

where $|z|=r<1$ and the inequality holds for equation (23) when $n=0$.

This completes the consequences of Theorem 3.2.

The consequences to Theorem 3.3 are Corollaries (3.8) -(3.11)

Corollary 4.8

A function $D^n f_j(z)\in A(\omega)$ is in the class $S^*\left(\omega,e^{\beta},0\right)$, if and only if

$$\sum_{k=p}^{\infty}\left(k+e^{\beta_j}-2\right)(r+d)^{k-1}a_k\le 1-e^{\beta_j} \tag{64}$$

where $|z|=r<1, |\omega|=d, n=0$

Corollary 4.9

Let $D^n f_j(z)\in A(\omega)$ belong to the class $S^*\left(\omega,e^{\beta},n\right)$

$$a_{k,j} \le \frac{1-e^{\beta_j}}{\psi_{0,m}\left(k+e^{\beta_j}-2\right)\left(r+d\right)^{k-1}}, k \ge 2 \tag{65}$$

Corollary 4.10

Given that

$D^n f_j(z) \in A(\omega)$ contained in $S^*\left(\omega, e^{\beta_j}, n\right)(\ j=1,\ldots,r)$ and $g_j(z) \in S^*\left(\omega, e^{\alpha_j}, n\right)(j=1,2,..,s)$ then

$D^n\left(f_1 * .\ .\ . * f_r\right)(z) \in S^*(w, \gamma, n)$

where

$$\gamma_j = 1 - \frac{(n-1)\left(1-e^{\beta_j}\right)^r\left(1-e^{\alpha_j}\right)^s}{n^s(r+d)^{k-1}\left(n+e^{\beta_j}-2\right)^r\left(n+e^{\alpha_j}-2\right)^s+\left(1-e^{\alpha}\right)^s\left(1-e^{\beta_j}\right)^r}\beta_j \tag{66}$$

Equation (66) is sharp for $D^n f_j(z)\ (j=1,\ldots,m)$ given by

$$D^n f_j(z) = (z-\omega) + \frac{1-e^{\beta}}{\left(k+kn+e^{\beta}-2\right)}(z-\omega)^k \tag{67}$$

Corollary 4.11

Suppose a function $D^n f(z) \in A(\omega)$ is contained in $S^*\left(\omega, e^{\beta}, 0\right)$, then

$D^n\left(f_1 * .\ .\ . * f_r\right)(z) \in S^*(w, \gamma, 0)$

where

$$\gamma = 1 - \frac{1-e^{\beta_j}}{\left(1-e^{\beta_j}\right)^m+\left(k+e^{\beta_j}-2\right)^m(r+d)^{(k-1)m}} \tag{68}$$

Equation (68) is sharp for $D^n f_j(z)\ (j=1,\ldots,m)$ given by

$$D^n f_j(z) = (z-\omega) + \frac{1-e^{\beta}}{\left(k+e^{\beta}-2\right)}(z-\omega)^k \tag{69}$$

This completes the consequences of Theorem 3.3.

5. Conclusion and Recommendations

This research work considered Hadamard product of holomorphic functions with arbitrary fixed point using Ruscheweyh derivative operator on some connected domain. The result obtained was validated with the existing results. This study showed that Ruscheweyh derivative is not a copied tool when the results of Hadamard product and coefficient inequalities are generalized. This research work also provides additional proofs for a class of holomorphic functions with arbitrary fixed point.

This research work extended the classes of univalent analytic functions studied by [12] using Ruscheweyh derivative operator. Larger classes of family are obtained. This study shows that Ruscheweyh derivative is a veritable tool when Hadamard product of holomorphic functions with arbitrary fixed point are to be generalized.

Also, this study extended the work of [15, 16] by providing additional proofs for a class of holomorphic functions with arbitrary fixed argument and fixed point using the generalized hypergeometric functions. This study shows that generalized hypergeometric functions are capable of transforming functions into Schwarz function. Ruscheweyh derivative operator for a generalized convex function and Starlike functions of order exponential β, given that. ω is a fixed point is specified.

Lastly, Hadamard product between a subclass of holomorphic functions with arbitrary fixed point and argument can be generalized using Ruscheweyh derivative.

Acknowledgements
The authors appreciate the effort of the independent readers and the constructive comments given by the reviewers.
Declaration of interest none

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