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Two-step Bi-basis Hybrid Block Method for Direct Approximation of Fourth Order Ordinary Differential Equations with Intermediate Bi-intra Points

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Abstract

Most recently, higher order problems are being addressed by decomposing it into system of lower order problems. However, it was discovered that methods with high order and strong stability were able to approximate the resulting systems accurately as the problems become unstable in the region of the new transformation field. This research actually sought for methods of solution of higher order problems without any need for system transformation. The method is proposed for the direct solution of fourth order ordinary differential equations. The fundamental basis is sought from the combination of Shifted Chebyshev Orthogonal Polynomial and the Hermite Orthogonal Polynomial, these polynomial functions are then used to obtain the method using the concept of interpolation and collocation. The proposed method is found to be consistent and zero-stable, which then implies convergence. From the numerical results obtained, the efficiency of the method was obtained and its superiority strength was also established when comparison was made with existing.

Keywords: Orthogonal, Hermite, interpolation and collocation, hybrid block method.

1. Introduction

Many problems resulting from applied sciences and engineering can be modelled in differential equations, one of such equations are the fourth order ordinary differential equations (ODEs) (Malek & Shekari, 2006). A practical example of problems in engineering is the problem of static deflection which a fourth order problem (Malek & Shekari, 2006 and Craig & Kurdila, 2006) The traditional and popular method of solving these equations is by transforming the equations into system of first order ODEs, but this process is tedious and time consuming when compared to direct methods (Waelah, Majid, Ismail & Suleiman, 2012 and Awoyemi 2005). The resulting systems project so many function evaluations which may result into a high noise as a result of the transformation. Over the years researchers have obtained direct methods to eradicate the transformation process and these new methods have proven effective over the traditional methods. However, the accuracy of the methods is needed to be improved on. These methods can be found in the the work of Kuboye and Omar (2015), Adeyeye and Omar (2019), Jator (2008), Yap and Ismail (2015), Abdelrahim and Omar (2017), Ndanusa, Adeboye, Mustapha \& Abdullahi (2020) and most recently Allogmany, Ismail, Abdul Majid and Ibrahim (2020) The work of Allogamy *et al.* (2020) was an implicit method which requires a predictor for obtaining some prior values. Our derivation was obtained using the approach of collocation and interpolation of the basis function. The execution time of our method is considerably small as it requires no predictor of prior values but the method obtained is a block which is an improvement on the accuracy of Allogmany *et al.* (2020).

2. Development of the Two-step Bi-basis Hybrid Block Method

For the solution of

$$y^{(iv)} = i(t, y', y'', y'''), \tag{1}$$

we sought an approximation of y(t) from the combination of a shifted chebyhev orthogonal polynomial and Hermite orthogonal polynomial which is expressed in the form:

$$y(t) \approx \sum_{r=0}^{p} a_r\, T_r^*(t) + \sum_{p+1}^{m} a_{p+1}\, H_{p+1}(t) \tag{2}$$

$$\text{taking } m = 8,\ p = \frac{m}{2}$$

where $T_r^*(t)$ and $H_r(t)$ are orthogonal polynomials of shifted chebyshev and Hermite. To obtain the coefficients of Eq. (1), we impose the following conditions:

$$y_r = y(t_r);\ r = 0;$$
$$f_{r+j} = y'^{(t_{r+j})};\ j = 0, \frac{1}{2}, 1, \frac{3}{2}, 2.$$
$$g_{r+j} = y''^{(r+j)};\ j = 0, \frac{1}{2}, 1, \frac{3}{2}, 2.$$
$$h_{r+j} = y'''^{(r+j)};\ j = 0, \frac{1}{2}, 1, \frac{3}{2}, 2.$$
$$i_{r+j} = y^{(iv)}(r+j);\ j = 0, \frac{1}{2}, 1, \frac{3}{2}, 2.$$

which produces a system of 9 equations in 9 unknowns. This equation was solved and the correspond coefficients were obtained. The resulting coefficients were then substituted in Eq. (2) and its corresponding derivatives to have the following block method:

$$y''_{r+\frac{1}{2}} = h^2 g_r + h^3 \frac{1}{2} h_r + h^4 \frac{367}{5760} i_r - h^4 \frac{47}{960} i_{n+1} - h^4 \frac{7}{1920} i_{n+2} + h^4 \frac{3}{32} h^4 i_{r+\frac{1}{2}} + h^4 \frac{29}{1440} i_{r+\frac{3}{2}}$$
$$y''_{r+1} = h^2 g_r + h^3 h_r + h^4 \frac{53}{360} i_r - h^4 \frac{1}{12} i_{n+1} - h^4 \frac{1}{120} i_{n+2} + h^4 \frac{2}{5} h^4 i_{r+\frac{1}{2}} + h^4 \frac{2}{45} i_{r+\frac{3}{2}}$$
$$y''_{r+\frac{3}{2}} = h^2 g_r + h^3 \frac{3}{2} h_r + h^4 \frac{147}{640} i_r + h^4 \frac{27}{320} i_{n+1} - h^4 \frac{9}{640} i_{n+2} + h^4 \frac{117}{160} h^4 i_{r+\frac{1}{2}} + h^4 \frac{3}{32} i_{r+\frac{3}{2}}$$
$$y''_{r+2} = h^2 g_r + 2h^3 h_r + h^4 \frac{14}{45} i_r + h^4 \frac{4}{15} i_{n+1} + h^4 \frac{16}{15} h^4 i_{r+\frac{1}{2}} + h^4 \frac{16}{45} i_{r+\frac{3}{2}}$$

$$y'''_{r+\frac{1}{2}} = h^3 h_r + h^4 \frac{251}{1440} i_r - h^4 \frac{11}{60} i_{n+1} - h^4 \frac{19}{1440} i_{n+2} + h^4 \frac{323}{720} h^4 i_{r+\frac{1}{2}} + h^4 \frac{53}{720} i_{r+\frac{3}{2}}$$
$$y'''_{r+1} = h^3 h_r + h^4 \frac{29}{180} i_r + h^4 \frac{2}{15} i_{n+1} - h^4 \frac{1}{180} i_{n+2} + h^4 \frac{31}{45} h^4 i_{r+\frac{1}{2}} + h^4 \frac{1}{45} i_{r+\frac{3}{2}}$$
$$y'''_{r+\frac{3}{2}} = h^3 h_r + h^4 \frac{27}{160} i_r + h^4 \frac{9}{20} i_{n+1} - h^4 \frac{3}{160} i_{n+2} + h^4 \frac{51}{80} h^4 i_{r+\frac{1}{2}} + h^4 \frac{21}{80} i_{r+\frac{3}{2}}$$
$$y'''_{r+2} = h^3 h_r + h^4 \frac{7}{45} i_r + h^4 \frac{4}{15} i_{n+1} + h^4 \frac{32}{45} h^4 i_{r+\frac{1}{2}} + h^4 \frac{32}{45} i_{r+\frac{3}{2}}$$

$$y_{r+\frac{1}{2}} = y_r + h \frac{1}{2} f_r + h^2 \frac{1}{8} g_r + h^3 \frac{1}{48} h_r + h^4 \frac{3373}{1935360} i_r - h^4 \frac{283}{322560} i_{r+1} - h^4 \frac{131}{1935360} i_{r+2} + h^4 \frac{139}{96768} i_{r+\frac{1}{2}} + h^4 \frac{179}{483840} i_{r+\frac{3}{2}}$$
$$y_{r+1} = y_r + h f_r + h^2 \frac{1}{2} g_r + h^3 \frac{1}{6} h_r + h^4 \frac{37}{1890} i_r - h^4 \frac{1}{72} i_{r+1} - \frac{1}{945} h^4 i_{r+2} + \frac{59}{1890} h^4 i_{r+\frac{1}{2}} + \frac{11}{3890} h^4 i_{r+\frac{3}{2}}$$
$$y_{r+\frac{3}{2}} = y_r + h \frac{3}{2} f_r + h^2 \frac{9}{8} g_r + h^3 \frac{9}{16} h_r + h^4 \frac{5319}{71680} i_r - h^4 \frac{1539}{35840} i_{n+1} - h^4 \frac{297}{71680} i_{n+2} + h^4 \frac{2889}{17920} h^4 i_{r+\frac{1}{2}} + h^4 \frac{81}{3584} i_{r+\frac{3}{2}}$$
$$y_{r+2} = y_r + 2h f_r + 2h^2 g_r + h^3 \frac{4}{3} h_r + h^4 \frac{176}{945} i_r - h^4 \frac{16}{315} i_{r+1} - \frac{2}{189} h^4 i_{r+2} + h^4 \frac{64}{135} i_{r+\frac{1}{2}} + h^4 \frac{64}{945} i_{r+\frac{3}{2}}$$

$$y'_{r+\frac{1}{2}} = h f_r + h^2 \frac{1}{2} g_r + h^3 \frac{1}{8} h_r + h^4 \frac{113}{8960} i_r - h^4 \frac{103}{13440} i_{n+1} - h^4 \frac{47}{80640} i_{n+2} + h^4 \frac{107}{8064} i_{r+\frac{1}{2}} + h^4 \frac{43}{13440} i_{r+\frac{3}{2}}$$
$$y'_{r+1} = h f_r + h^2 g_r + h^3 \frac{1}{2} h_r + h^4 \frac{331}{5040} i_r - h^4 \frac{1}{21} i_{n+1} - h^4 \frac{19}{5040} i_{n+2} + h^4 \frac{83}{630} i_{r+\frac{1}{2}} + h^4 \frac{23}{630} i_{r+\frac{3}{2}}$$
$$y'_{r+\frac{3}{2}} = h f_r + h^2 \frac{3}{2} g_r + h^3 \frac{9}{8} h_r + h^4 \frac{1431}{8960} i_r - h^4 \frac{243}{4480} i_{n+1} - h^4 \frac{81}{8960} i_{n+2} + h^4 \frac{1863}{4480} i_{r+\frac{1}{2}} + h^4 \frac{45}{896} i_{r+\frac{3}{2}}$$
$$y'_{r+2} = h f_r + 2h^2 g_r + 2h^3 h_r + h^4 \frac{31}{105} i_r + h^4 \frac{4}{105} i_{n+1} - h^4 \frac{1}{63} i_{n+2} + \frac{272}{315} h^4 i_{r+\frac{1}{2}} + h^4 \frac{16}{105} i_{r+\frac{3}{2}}$$

2.1. Analysis of the Method

Order and Consistency

The block methods are expanded in Taylor series, related terms were collected and thus we have the order and corresponding error constant as given below:

Table 1: Table of Order for each Formula of and Corresponding Error Constant

S/N	Formula	Order	Error Constant
1	$y_{r+\frac{1}{2}}$	4	$\frac{-53}{46080}$
2	y_{r+1}	4	$\frac{-23}{1260}$
3	$y_{r+\frac{3}{2}}$	4	$\frac{-351}{7168}$
4	y_{r+2}	4	$\frac{4}{315}$
5	$y'_{r+\frac{1}{2}}$	4	$\frac{-401}{40320}$
6	y'_{r+1}	4	$\frac{-31}{504}$
7	$y'_{r+\frac{3}{2}}$	4	$\frac{-117}{4480}$
8	y'_{r+2}	4	$\frac{22}{63}$
9	$y''_{r+\frac{1}{2}}$	4	$\frac{-187}{2880}$
10	y''_{r+1}	4	$\frac{-17}{180}$
11	$y''_{r+\frac{3}{2}}$	4	$\frac{21}{64}$
12	y''_{r+2}	4	$\frac{56}{45}$
13	$y'''_{r+\frac{1}{2}}$	4	$\frac{-59}{240}$
14	y'''_{r+1}	4	$\frac{-1}{5}$
15	$y'''_{r+\frac{3}{2}}$	4	$\frac{31}{80}$
16	y'''_{r+2}	4	$\frac{23}{30}$

The proposed is concluded to be simply consistence, this is according to Lambert which presented that the necessary and sufficient condition for a numerical scheme to be consistence is for it to have order of at least 1.

Zero-stability

This is a specification related to the method as h tends to zero. Taking the limit as h→ 0 in the block method we have the system of equations as below:

$$y_{r+\frac{1}{2}} = y_r,\ y_{r+1} = y_r,\ y_{r+\frac{3}{2}} = y_r,\ y_{r+2} = y_r$$
$$y'_{r+\frac{1}{2}} = y_r,\ y'_{r+1} = y_r,\ y'_{r+\frac{3}{2}} = y_r,\ y'_{r+2} = y_r$$
$$y''_{r+\frac{1}{2}} = y_r,\ y''_{r+1} = y_r,\ y''_{r+\frac{3}{2}} = y_r,\ y''_{r+2} = y_r$$
$$y'''_{r+\frac{1}{2}} = y_r,\ y'''_{r+1} = y_r,\ y'''_{r+\frac{3}{2}} = y_r,\ y'''_{r+2} = y_r$$

which can be written as:

$$IY_r - A_0 Y_{r-1} = 0$$

where:

$$I = Identity\ matrix\ of\ order\ 16.$$

$$A_0 = \begin{pmatrix} 0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1 \end{pmatrix},\ Y_r = \begin{pmatrix} y_{r+\frac{1}{2}} \\ y_{r+1} \\ y_{r+\frac{3}{2}} \\ y_{r+2} \\ y'_{r+\frac{1}{2}} \\ y'_{r+1} \\ y'_{r+\frac{3}{2}} \\ y'_{r+2} \\ y''_{r+\frac{1}{2}} \\ y''_{r+1} \\ y''_{r+\frac{3}{2}} \\ y''_{r+2} \\ y'''_{r+\frac{1}{2}} \\ y'''_{r+1} \\ y'''_{r+\frac{3}{2}} \\ y'''_{r+2} \end{pmatrix},\ Y_{r-1} = \begin{pmatrix} y_r \\ y_r \\ y_r \\ y_r \\ y_r \\ y_r \\ y_r \\ y_r \\ y_r \\ y_r \\ y_r \\ y_r \\ y_r \\ y_r \\ y_r \\ y_r \end{pmatrix}$$

A block method is zero-stable if the roots r_i of the first characteristic polynomial $\xi(r) = |Ir - A_0| \leq 1$. From the block method, we obtain the first characteristic polynomial as

$$r^8(r-1)^8 = 0$$

and thus, obtaining the roots as $r = 0,0,0,0,0,0,0,0,1,1,1,1,1,1,1,1$ in which none exceed 1. We then simply conclude that the proposed method is zero-stable.

3. Numerical Implementation of the Proposed Method on some Selected Examples

This section contains the numerical examples considered and tables of results for the proposed method in comparison with Kuboye & Omar (2015), Jator, Yap & Ismail (2008), Abdelrahim & Omar (2017) and Allogmany *et al.* (2020). Throughout this report, Absolute error is taken to be the absolute difference between the exact solution and a numerical method; Time, measured in $seconds$, is taken to be the time of execution of a method.

Example 1

$$y^{(4)}(t) = -y''(t),\ \ 0 \leq t \leq \frac{\pi}{2},$$
$$y(0) = 0,$$
$$y'(0) = \frac{-1.1}{72 - 50\pi},$$
$$y''(0) = \frac{1}{144 - 100\pi},$$
$$y'''(0) = \frac{1.2}{144 - 100\pi},$$

Exact solution:

$$y(t) = \frac{1 - t - cos(t) - 1.2sin(t)}{144 - 100\pi}$$

Example 2

$$y^{(4)}(t) = y''' + y'' + y' + 2y,\ t \in [0,2]$$
$$y(0) = y'(0) = y''(0) = 0,$$
$$y'''(0) = 30.$$

Exact solution:

$$y(t) = 2\exp(2t) - 5\exp(-t) + 3cos(t) - 9sin(t)$$

Example 3

We consider the Ship Dynamic Problem:

$$y^{(4)}(t) = -3y'' - y\big(2 + \epsilon cos(\omega t)\big),\ \ t > 0,$$
$$y(0) = 1, y'(0) = y''(0) = y'''(0),$$

where $\epsilon = 0$, the theoretical solution is:

$$y(t) = 2cos(t) - cos\big(\sqrt{2}t\big).$$

Table of Results

Table 2: Table of Absolute Errors for Different Methods on Example 1, $h = 0.01$.

t	Kuboye & Omar	Allogmany *et al.*	Present Method
0.01	5.421011 -20	5.420312 -20	5.582625 -21
0.02	5.421011 -20	5.419616 -20	1.290732 -21
0.03	2.710505 -19	2.709462 -19	2.582165 -20
0.04	1.084202 -19	0.000000 +00	3.873011 -20
0.05	3.252607 -19	3.250530 -19	5.163265 -20
0.06	3.252607 -19	3.250120 -19	6.452929 -20
0.08	1.734723 -18	0.000000 +00	9.030479 -20
0.09	4.336809 -18	2.165937 -19	1.031836 -20
0.10	8.456777 -18	0.000000 +00	1.160566 -20
Time (s)	NA	NA	0.1563

Table 3: Table of Absolute Errors for Different Methods on Example 2

Step Number	Jator	Yap & Ismail	Abdelrahim & Omar	Allogmany et al.	Present Method
20	1.25612 -04	1.74000 -08	8.07000 -10	1.851538 -10	9.042072 -11
40	1.90752 -06	8.45000 -11	3.22000 -12	2.328543 -12	8.270990 -13
80	2.96411 -08	3.69000 -13	-	2.115566 -14	7.897214 -15
100	8.65117 -09	7.11000 -14	-	1.079081 -14	7.533771 -15
Time (s)	NA	NA	NA	NA	0.2500

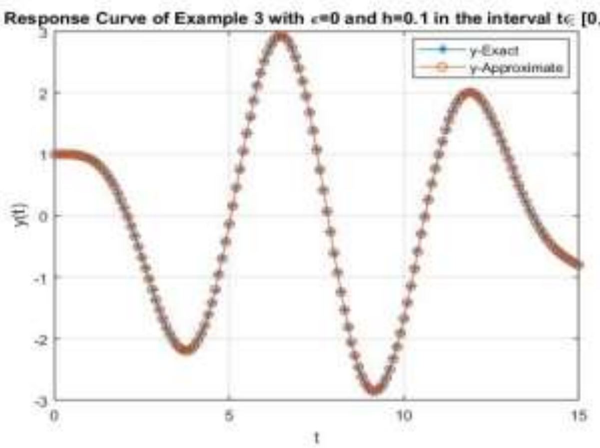

Figure 1: Response Curve of Example 3 with ϵ=0 and $h = 0.1$ in the interval $t \in [0,15]$

4. Discussion of Results and Conclusion

Numerical method was programmed in MATLAB 9.2 version on a personal computer. An extensive comparison of the absolute errors was carried out for the method of Kuboye & Omar (2015), Jator, Yap & Ismail (2008), Abdelrahim & Omar (2017), Allogmany *et al.* (2020), and the proposed method in Tables 1 and 3. The step size h was taken to be 0.01 for example 1 and results presented in the interval t ∈ [0.01,0.1] while the number of iterations or the step numbers were taken to be 20, 40, 80 and 100 in example 2. The execution time (measured in Seconds) of the proposed method was also reported. The numerical results of the proposed method established the efficiency and superiority, in-term of accuracy, over existing methods found in literature.

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Declaration of Competing Interest:

The authors declare that there is no conflict-of-interest null competing interest. All the authors contributed in the preparation of this paper.

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